



# Quiver Matrix Mechanics for IIB String Theory (I): Wrapping Membranes and Emergent Dimension

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## Abstract

In this paper we present a discrete, non-perturbative formulation for type IIB string theory. Being a supersymmetric quiver matrix mechanics model in the framework of M(atr)ix theory, it is a generalization of our previous proposal of compactification via orbifolding for deconstructed IIA strings. In the continuum limit, our matrix mechanics becomes a  $(2 + 1)$ -dimensional Yang-Mills theory with 16 supercharges. At the discrete level, we are able to construct explicitly the solitonic states that correspond to membranes wrapping on the compactified torus in target space. These states have a manifestly  $SL(2, \mathbb{Z})$ -invariant spectrum with correct membrane tension, and give rise to an emergent flat dimension when the compactified torus shrinks to vanishing size.

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# 1 Introduction

In the string/M-theory duality web, both the IIA/M and IIB/M duality conjectures involve the emergence of a new spatial dimension in a certain limit. M-theory, according to our current understanding, is the 11-dimensional dual of the strongly coupled 10-dimensional IIA string theory [1]. Here a new dimension emerges, because the D-particles, carrying RR charges, can be viewed as the Kaluza-Klein momentum modes along a circle in the “hidden eleventh” dimension. The size of the circle depends on the IIA string coupling in such a way that when the string coupling tends to infinity, the size of the circle also becomes infinite. In other words, IIA string theory should be dual to M-theory compactified on a circle, and the hidden dimension opens up as a new flat dimension in the strong-coupling limit.

For IIB/M duality, the story is obviously more complicated [2, 3, 4]. IIB string theory is conjectured to be dual to M-theory compactified on a 2-torus. The IIB string coupling depends on the area of the 2-torus, in such a way that the weak coupled IIB string theory should be recovered when the area of the hidden 2-torus shrinks to zero. A great advantage of this scenario is that the mysterious  $SL(2, \mathcal{Z})$  duality in IIB string theory can be interpreted as the geometric, modular transformation of the hidden 2-torus. But wait, there is apparently a mismatch of dimensionality here: with the compactified two dimensions diminished in M theory, it seems that only eight spatial dimensions remain, but IIB string theory lives in nine, not eight, dimensional space. The clever solution [2] to this puzzle is that in M theory there are topological soliton-like states, corresponding to wrapping a membrane around the compactified 2-torus. Because of the conjectured  $SL(2, \mathcal{Z})$  invariance, the energy of the wrapping membrane should only depend on its wrapping number  $w$ . With the torus shrinking to zero, a tower of the wrapping membrane states labeled by  $w$  gives rise to the momentum states in a new, flat, emergent dimension.

The picture looks perfect. But can we really formulate it in a mathematical for-

malism with above-mentioned ideas demonstrated explicitly? Certainly this needs a non-perturbative formulation of M-theory. Though a fully Lorentz invariant formulation of M-theory is not available yet, fortunately we have had a promising candidate (or conjecture), namely the BFSS M(atrrix) Theory [5], in a particular kinematical limit, the infinite momentum frame (IMF), or in the discrete light-cone quantization (DLCQ) [6]. It was shown that M(atrrix) Theory compactified on an  $n$ -torus (for  $n \leq 3$ ) is just an  $(n+1)$ -dimensional supersymmetric Yang-Mills (SYM) theory [7]. Facilitated by this fact, the IIA/M duality was verified soon after BFSS conjecture. Namely M(atrrix) Theory compactified on a circle was explicitly shown, via the so-called “9-11 flip”, to give rise to a description of (second quantized) IIA strings [8]. As to IIB/M duality, again the story was not so simple as IIA/M duality. The statement that M(atrrix) theory compactified on a 2-torus (with one side in the longitudinal direction) is dual to IIB string theory compactified on a circle was checked [9] for the action of  $D$ -string [10] and the energy of  $(p, q)$ -strings etc. An emergent dimension with full  $O(8)$  invariance together with seven ordinary dimensions were argued to come about in the  $(2+1)$ -dimensional SYM that results from compactifying M(atrrix) Theory on a transverse torus [11, 12, 13]. These arguments invoked the conjectured  $SL(2, \mathcal{Z})$  duality of  $(3+1)$ -dimensional SYM. However, the explicit construction of wrapping membrane states and a proof of the  $SL(2, \mathcal{Z})$  invariance of these states, which plays a pivotal role in the IIB/M duality, are still lacking.

Here let us try to scrutinize the origin of the difficulty for constructing a wrapping membrane state in M(atrrix) theory. First recall that in the usual membrane theory, a nonlinear sigma model with continuous world-volume, a membrane state wrapping on a compactified 2-torus is formulated by

$$\begin{aligned} x^1(p, q) &= R_1(mp + nq) + (\text{oscillator modes}), \\ x^2(p, q) &= R_2(sp + tq) + (\text{oscillator modes}) \end{aligned} \tag{1}$$

where  $p$  and  $q$  are two affine parameters of the membrane, two independent cycles on the compactified 2-torus have  $R_1$ ,  $R_2$  as their sizes and  $x_1$ ,  $x_2$  their affine parameters,  $m, n, s, t$  four integers characterizing *linearly* how the membrane wraps on the torus. However, in M(atric) Theory the target space coordinates are lifted to matrices, and the basis of the membrane world-volume functions,  $e^{ip}$  and  $e^{iq}$ , are transcribed to noncommutative, (so-called) clock and shift matrices  $U$  and  $V$  of finite rank. Moreover, when the theory is compactified on a torus, the quotient conditions for compactification require the matrix coordinates of the target space to become covariant derivatives on the *dual torus*. Thus, the difficulty in constructing the matrix analog of the wrapping membrane states in Eq. (1) lies in how to extract the affine parameters  $p, q$  from their finite matrix counterparts  $U$  and  $V$ , as well as to keep the linear relation containing the information of wrapping in Eq. (1). In a word, the problem is a mismatch between the linear coordinates of the target torus and the nonlinear coordinates of the membrane degrees of freedom. To our knowledge, no explicit construction to overcome this mismatch is known in the literature. The main goal of this paper is to fill this gap, namely we want to develop a M(atric) Theory description of IIB string theory, which makes IIB/M duality and  $SL(2, \mathcal{Z})$ -duality more accessible to analytic treatments.

Our strategy to change the mismatch to be a match is to simply let both sides to be nonlinear; technically, we *deconstruct* the  $(2 + 1)$ -dimensional SYM that results from compactifying M(atric) Theory on a torus and then, within the resulting framework of quiver matrix mechanics, to construct the matrix membrane wrapping states. In our previous paper [14], we have been able to deconstruct the Matrix String Theory for type IIA strings, which is a  $(1 + 1)$ -dimensional SYM. Here we generalize this deconstruction to type IIB strings. The basic idea is to achieve compactification by orbifolding the M(atric) Theory, like in Ref. [14]. There, to compactify the theory on a circle, we took the orbifolding (or quotient) group to

be  $\mathcal{Z}_N$ ; but here to compactify the theory on a 2-torus we need to take a quotient with the group  $\mathcal{Z}_N \otimes \mathcal{Z}_N$  (or shortly  $\mathcal{Z}_N^2$ ). This leads to a supersymmetric quiver matrix mechanics with a product gauge group and bi-fundamental matter. By assigning non-zero vacuum expectation values (VEV) to bi-fundamental scalars, the quiver theory looks like a theory on a planar triangular lattice, whose continuum limit gives rise to the desired  $(2 + 1)$ -dimensional SYM. From the point of view of D-particles in target space, this is nothing but “compactifying via orbifolding”, or “deconstruction of compactification”. (The use of the literary term and analytic technique of “deconstruction” in high-energy physics was advocated in [15]. Similar applications of the deconstruction method can also be found in [16] [17][18] [19][20][21].) It is in this approach we have been able to accomplish a down-to-earth construction of wrapping membrane states in M(atric) Theory, a modest step to better understand IIB/M duality.

In this paper we will carry out our deconstruction for the simplest case with a right triangular lattice, and concentrate on the construction of the membrane states. The general case with a slanted triangular lattice and a detailed study of  $SL(2, \mathcal{Z})$  duality are left to the subsequent paper(s). We organize this paper as follows. In Sec. 2, we will give a brief review of deconstruction of the Matrix String Theory, to stipulate our notations and demonstrate the idea to approximate a compactification by an orbifolding sequence (in the so-called theory space). Then in Sec. 3, extending this approach, we achieve the deconstruction of the  $(2 + 1)$ -dimensional SYM that describes IIB strings by means of orbifolding M(atric) Theory by  $\mathcal{Z}_N \otimes \mathcal{Z}_N$ , resulting in a supersymmetric quiver matrix mechanics. Metric aspects of this de(construction) after assigning VEV’s is explored in Sec. 4; then the desired SYM is shown to emerge in the continuum limit with  $N \rightarrow \infty$  upon choosing the simplest (right) triangular lattice. Sec. 5 is devoted to the central issue of this paper, namely explicit construction of topologically non-trivial states in quiver

mechanics that in the continuum limit give rise to the membranes wrapping on the compactified torus. The energy spectrum of these states is shown to depend only on the  $SL(2, \mathcal{Z})$ -invariant wrapping number  $w$ . This provides an explicit verification of  $SL(2, \mathcal{Z})$  invariance, as well as the correct membrane tension in our discrete approach. Moreover, the spectrum of these states is such that it can be identified as the spectrum of momentum states in an emergent flat transverse dimension when the size of the compactified torus shrinks to zero. Finally in Sec. 6 discussions and perspectives are presented.

## 2 A Brief Review: Matrix String (De)Construction

This preparative section serves two purposes: to establish our notations and to review briefly the physical aspects of our previous (de)construction of the Matrix String Theory.

### 2.1 Preliminaries and Notations

By now it is well-known that at strong couplings of IIA string theory, a new spatial dimension emerges so that the 10-dimensional IIA theory is dual to an 11-dimensional theory, dubbed the name M-theory [1]. In accordance with the M(atr)ix Theory conjecture [5] and as shown by Seiberg [22] and Sen [23], the same dynamics that governs  $N$  low-energy D-particles in a ten-dimensional Minkowski space-time actually captures all the information of M-theory in the discrete light-cone quantization. If we label the spatial coordinates by  $y^1, y^2, \dots, y^9$ , then the basic idea of M(atr)ix Theory is that, to incorporate open strings stretched between  $N$  D-particles, one has to lift the D-particle coordinates to  $N$ -by- $N$  matrices,  $Y^I$  ( $I = 1, \dots, 9$ ), as the basic dynamical variables.

Formally the Matrix Theory *a la* BFSS can be formulated as a dimensional



reduction of  $d = 9 + 1$ ,  $\mathcal{N} = 1$  supersymmetric Yang-Mills theory:

$$S = \int d^{10}x \left\{ -\frac{1}{4g^2} \text{Tr} F_{MN} F^{MN} - \frac{i}{2g^2} \text{Tr} \lambda^T \Gamma^0 \Gamma^M D_M \lambda \right\}$$

where the Majorana-Weyl spinor  $\lambda$ , whose components will be labeled as  $\lambda^{s_0 s_1 s_2 s_3}$  ( $s_a = 1, 2, a = 0, 1, 2, 3$ ), satisfy the spinor constraint equations  $\lambda = -\Gamma^{11} \lambda$ ,  $\lambda^* = \lambda$ . The dimensional reduction follows the procedure below:  $\int d^{10}x \rightarrow \int dt V_9$ ;  $\tilde{g}^2 := g^2/V_9$ ; then  $A^I =: -X^I/\alpha'$ ,  $\tilde{g}^2 \alpha'^2 = g_s \alpha'^{1/2}$ ,  $\lambda^2/\tilde{g}^2 \rightarrow -\lambda^2$  and  $X^I =: g_s^{1/3} \alpha'^{1/2} Y^I$  where  $I = 1, \dots, 9$ . Note that in accordance with the IIA/M duality, the M-theory parameters are related to those of IIA string theory by  $R = g_s l_s$ ,  $l_p = g_s^{1/3} l_s$ , in which  $l_p$  is the eleven-dimensional Planck length and  $R$  the radius of the hidden M-circle; In our convention,  $\alpha' = l_s^2$ . Time variable and M-circle radius can be rescaled in units of Planck length to become dimensionless:  $t = l_p \tau$ ,  $R = R_{11} l_p$ , respectively.

After all these efforts, the BFSS action reads

$$S = \int d\tau \text{Tr} \left\{ \frac{1}{2R_{11}} [D_\tau, Y^I]^2 + \frac{R_{11}}{4} [Y^I, Y^J]^2 - \frac{i}{2} \lambda^T [D_\tau, \lambda] + \frac{R_{11}}{2} \lambda^T \gamma^I [Y_I, \lambda] \right\}, \quad (2)$$

in which eleven-dimensional Planck length is taken to be unity,  $D_\tau = d/d\tau - i[Y^0, \cdot]$  is the covariant time derivative, and  $Y^0$  is the gauge connection in the temporal direction. A representation of the gamma matrices  $\gamma^I$  in Eq. (2) is listed below for later uses:

$$\begin{aligned} \gamma^0 &:= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \quad , \quad \gamma^1 = \epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon, \\ \gamma^2 &= \tau_1 \otimes \mathbf{1} \otimes \epsilon \otimes \epsilon \quad , \quad \gamma^3 = \tau_3 \otimes \mathbf{1} \otimes \epsilon \otimes \epsilon, \\ \gamma^4 &= \epsilon \otimes \tau_1 \otimes \mathbf{1} \otimes \epsilon \quad , \quad \gamma^5 = \epsilon \otimes \tau_3 \otimes \mathbf{1} \otimes \epsilon, \\ \gamma^6 &= \mathbf{1} \otimes \epsilon \otimes \tau_1 \otimes \epsilon \quad , \quad \gamma^7 = \mathbf{1} \otimes \epsilon \otimes \tau_3 \otimes \epsilon, \\ \gamma^8 &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \tau_1 \quad , \quad \gamma^9 = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \tau_3, \end{aligned} \quad (3)$$

where  $\epsilon = i\tau_2$  and  $\tau_a$  ( $a = 1, 2, 3$ ) are Pauli matrices.



In the following, the  $N$ -by- $N$  clock and shift matrices will be used frequently. We denote them by  $U_N = \text{diag}(\omega_N, \omega_N^2, \dots, \omega_N^N)$ , where  $\omega_N = e^{i2\pi/N}$ , and

$$V_N := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \quad (4)$$

Quite often we will just write  $U, V$  when the rank  $N$  is understood.

## 2.2 Compactification via Orbifolding

In this subsection we give a brief review of our previous work [14], to show how the Matrix Theory compactification on a circle was achieved via orbifolding and how the worldsheet/target-space duality emerges in a natural manner.

Consider a group of  $K$  D-particles moving in a  $\mathcal{C}^2/\mathcal{Z}_N$  orbifold background; the orbifolding is performed via equivalence relations  $z^1 \sim e^{i2\pi/N} z^1$ ,  $z^2 \sim e^{-i2\pi/N} z^2$ , where  $z^1 = (y^6 + iy^7)/\sqrt{2}$ ,  $z^2 = (y^8 + iy^9)/\sqrt{2}$ . The complex coordinates  $z^1, z^2$  can be parameterized in a polar coordinate form,  $z^1 = \rho_1 e^{i(\vartheta+\varphi)}/\sqrt{2}$ ,  $z^2 = \rho_2 e^{i(\vartheta-\varphi)}/\sqrt{2}$ ; the quotient conditions, in this parametrization, are simply  $\varphi \sim \varphi + 2\pi/N$ . To consider  $K$  D-particles on the orbifold, we must lift the D-particle coordinate to  $KN$ -by- $KN$  matrices, for which we will use upper-cased letters  $Z^1, Z^2$ . To satisfy the orbifolding conditions these matrices are of the special form  $z^a V_N$ , where  $z^a$  ( $a = 1, 2$ ) are in the form of  $N \times N$  block-diagonal matrix with each block being a  $K$ -by- $K$  matrix and the unities in  $V_N$  (see Eq. (4)) understood as  $\mathbf{1}_K$ . Besides orbifolding, we push the D-particles away from the singularity by assigning nonzero VEV to  $z^1$  and  $z^2$ . Note that the moduli of the D-particle coordinates,  $\langle z^1 \rangle = c_1/\sqrt{2}$ ,  $\langle z^2 \rangle = c_2/\sqrt{2}$ , can be rotated to  $\langle z^{1'} \rangle = c'_1/\sqrt{2}$ ,  $c'_1 = \sqrt{|c_1|^2 + |c_2|^2}$  with  $\langle z^{2'} \rangle = 0$ , by an element of  $SU(2)$  acting on the  $\mathcal{C}^2$ . (The orbifolding results in a circular quiver





diagram with  $N$  sites. The  $Z$ -fields live on the links connecting neighboring sites, indicated by the presence of the block-matrix  $V_N$ .) So essentially only one modulus is needed to characterize the VEV.

This modulus can be interpreted in two different pictures:

$$\text{I: } 2\pi c'_1 = NL, \quad (5)$$

$$\text{II: } N/c'_1 = 2\pi\Sigma. \quad (6)$$

On the one hand,  $L$  is interpreted as the size of the circle on the orbifold in the target space, which is orthogonal to the radial direction and at the radius  $c'_1$ . Viewed from the target space, D-particles live around this circle, so it can be viewed as the compactified circle, or the  $M$ -circle for IIA theory. On the other hand,  $\Sigma$  in Eq. (5) can be viewed as the size of the circle that is dual to the target circle  $L$ . It is both amazing and amusing to see that with the fluctuations in  $Z^1$  and  $Z^2$  included, the orbifolded BFSS action can be equivalently written as an action on a one-dimensional lattice with lattice constant  $a = 2\pi\Sigma/N$ . Furthermore, in the limit  $N \rightarrow \infty$ , this quiver mechanics action approaches to the  $d = 1+1$  SYM (on the dual circle  $\Sigma$ ) with 16 supercharges, known to describe the Matrix String theory, which was previously obtained by compactifying M(atrix) Theory on the circle  $L$  directly. (Note that the so-called “9-11 flip”, that was necessary to give an appropriate interpretation for the compactified Matrix Theory on  $L$ , is not necessary at all in the orbifolding context).

Thus, what we have done is actually to (de)construct the (type IIA) string worldsheet via a circular quiver diagram, and the relations (5) and (6) between the geometric parameters is a clear demonstration of the worldsheet/target-space duality: Namely Eq. (5) is over target space orbifold, in which  $c'_1$  prescribes the location of the  $M$ -circle; Eq. (6) is over worldsheet cycle, in which  $c'_1$  is the inverse of the lattice constant. The dual relation between the two interpretations and the compactification scale is shown in Figure 1.

The methodology underlying the above deconstruction of IIA/M duality and the

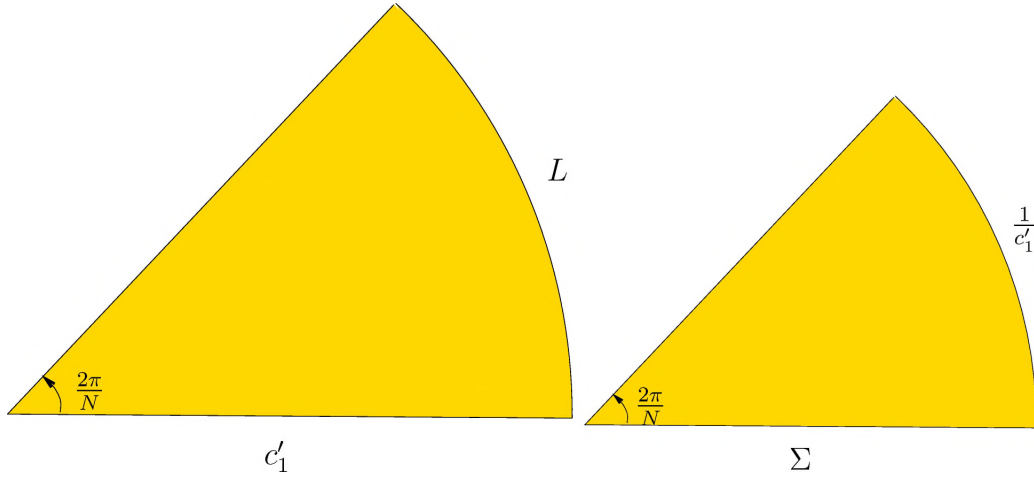


Figure 1: Worldsheet/target-space duality in  $\mathcal{C}^2/\mathcal{Z}_N$  Orbifold

relationship between the Matrix Theory, type II string theory and our quiver quantum mechanics are demonstrated in a commuting diagram (Figure 2). In subsequent sections we will show that this diagram applies to the deconstruction of IIB strings as well. Additional complications from higher dimensions will emerge; and we will show how to deal with them in this and sequential paper.

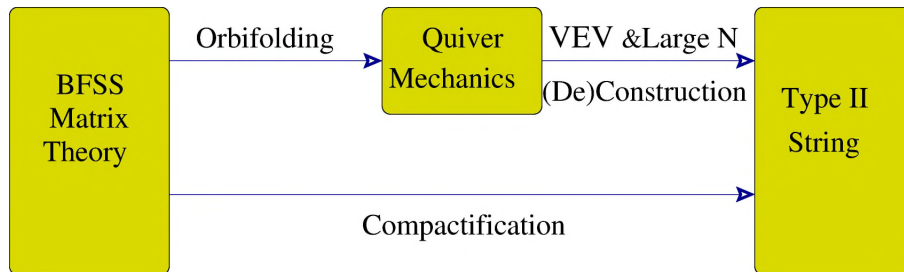


Figure 2: Relation between Quiver Mechanics and Matrix Theory, Type II String Theory



### 3 $\mathcal{Z}_N \otimes \mathcal{Z}_N$ Orbifolded Matrix Theory

Now we proceed to make and justify our proposal that the  $\mathcal{Z}_N \otimes \mathcal{Z}_N$  orbifolded M(atric) Theory, being a supersymmetric quiver matrix mechanics, provides a promising candidate for the nonperturbative formulation of IIB string theory.

Consider the orbifold  $\mathcal{C}^3/\mathcal{Z}_N \otimes \mathcal{Z}_N$ , where the complex coordinates of  $\mathcal{C}^3$  are  $z^a = (y^{2a+2} + iy^{2a+3})/\sqrt{2}$  for  $a = 1, 2, 3$  and the quotient conditions for orbifolding are

$$\text{I: } z^1 \sim e^{-i2\pi/N} z^1, z^2 \sim e^{i2\pi/N} z^2, z^3 \sim z^3; \quad (7)$$

$$\text{II: } z^1 \sim e^{-i2\pi/N} z^1, z^2 \sim z^2, z^3 \sim e^{i2\pi/N} z^3. \quad (8)$$

As did in the last section, we lift the D-particle coordinates to  $KN^2$ -by- $KN^2$  matrix variables  $Z^a = (Y^{2a+2} + iY^{2a+3})/\sqrt{2}$ , ( $a = 1, 2, 3$ ), and subject them to the following orbifolding conditions:

$$\hat{U}_1^\dagger Z^a \hat{U}_1 = \omega_N^{M_{45}-M_{67}} Z^a, \quad \hat{U}_2^\dagger Z^a \hat{U}_2 = \omega_N^{M_{45}-M_{89}} Z^a, \quad (9)$$

in which  $M_{IJ}$  are rotational generators on  $IJ$ -plane for vectors in nine-dimensional transverse space, and  $\hat{U}_1 := \mathbf{1}_K \otimes U_N \otimes \mathbf{1}_N$ ,  $\hat{U}_2 := \mathbf{1}_K \otimes \mathbf{1}_N \otimes U_N$  embed the action of rotations into the gauge group,  $U(KN^2)$ , of M(atric) Theory.

Similarly introduce complexified fermionic coordinates for the Majorana-Weyl spinor  $\lambda$  in M(atric) Theory. For each real spinor index  $s = 1, 2$  for  $\lambda$ , introduce complex spinor index  $t = \pm$  through

$$\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} = T \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix}, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

Therefore,  $(\lambda^t)^\dagger = \lambda^{-t}$ . In addition,

$$T^\dagger \tau_1 T = -\tau_2, T^\dagger \tau_2 T = -\tau_3, T^\dagger \tau_3 T = \tau_1.$$



After using this  $T$  to change the basis for the last three spinor indices for fermionic coordinates, the gamma matrices become

$$\begin{aligned}\gamma^0 &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \quad , \quad \gamma^1 = -\tau_2 \otimes \tau_3 \otimes \tau_3 \otimes \tau_3, \\ \gamma^2 &= -\tau_1 \otimes \mathbf{1} \otimes \tau_3 \otimes \tau_3 \quad , \quad \gamma^3 = -\tau_3 \otimes \mathbf{1} \otimes \tau_3 \otimes \tau_3, \\ \gamma^4 &= -\tau_2 \otimes \tau_2 \otimes \mathbf{1} \otimes \tau_3 \quad , \quad \gamma^5 = \tau_2 \otimes \tau_1 \otimes \mathbf{1} \otimes \tau_3, \\ \gamma^6 &= \mathbf{1} \otimes \tau_3 \otimes \tau_2 \otimes \tau_3 \quad , \quad \gamma^7 = -\mathbf{1} \otimes \tau_3 \otimes \tau_1 \otimes \tau_3, \\ \gamma^8 &= -\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \tau_2 \quad , \quad \gamma^9 = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \tau_1.\end{aligned}\tag{10}$$

The sixteen-component fermionic coordinate  $\lambda$  is thus complexified, as an eight-component complex spinor. The orbifolding conditions for fermionic variables read

$$\hat{U}_1^\dagger \lambda \hat{U}_1 = \omega^{\sigma_{45} - \sigma_{67}} \lambda \quad , \quad \hat{U}_2^\dagger \lambda \hat{U}_2 = \omega^{\sigma_{45} - \sigma_{89}} \lambda,\tag{11}$$

where  $\sigma_{IJ} = i[\gamma^J, \gamma^I]/4$  are the rotational generators for spinors:

$$\begin{aligned}\sigma^{45} &= \frac{1}{2} \mathbf{1} \otimes \tau_3 \otimes \mathbf{1} \otimes \mathbf{1}, \\ \sigma^{67} &= \frac{1}{2} \mathbf{1} \otimes \mathbf{1} \otimes \tau_3 \otimes \mathbf{1}, \\ \sigma^{89} &= \frac{1}{2} \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \tau_3.\end{aligned}\tag{12}$$

Quantum numbers of each variable are summarized in Table 1. In this table,  $J_{IJ} = M_{IJ}$  for vectors,  $J_{IJ} = \sigma_{IJ}$  for spinors, and remember that  $(\lambda^{s_0, t_1, t_2, t_3})^\dagger = \lambda^{s_0, -t_1, -t_2, -t_3}$ .

Schematically, the field contents of our orbifolded M(atrix) Theory is encoded by a quiver diagram in the form of a planar triangular lattice. Figure 3 illustrates six cells in the quiver diagram. Each variable is a  $KN^2$ -by- $KN^2$  matrix, which can also be viewed as a  $K$ -by- $K$  matrix field living on the sites or links in the quiver diagram with  $N^2$  sites. The matrix block form for the  $N^2$ -by- $N^2$  indices of these variables is dictated by the quiver diagram. From either Table 1 or Figure 3, we can read off that  $Y^{0,1,2,3}$  and  $\lambda^{s_0+++}$ , hence  $\lambda^{s_0---}$ , are defined on the sites, while



Table 1: Quantum Numbers of Variables in the BFSS Matrix Theory upon Orbifolding

	$J_{45}$	$J_{67}$	$J_{89}$	$J_{45} - J_{67}$	$J_{45} - J_{89}$
$Y^{0,1,2,3}$	0	0	0	0	0
$Z^1$	-1	0	0	-1	-1
$Z^2$	0	-1	0	1	0
$Z^3$	0	0	-1	0	1
$\lambda^{s_0+++}$	1/2	1/2	1/2	0	0
$\lambda^{s_0++-}$	1/2	1/2	-1/2	0	1
$\lambda^{s_0+-+}$	1/2	-1/2	1/2	1	0
$\lambda^{s_0-++}$	-1/2	1/2	1/2	-1	-1

$Z^{1,2,3}$  and other fermionic coordinates, as well as their hermitian conjugate, on the links [19]. In the following, we will often suppress the  $K$ -by- $K$  indices, while using a pair of integers  $(m, n)$ ,  $m, n = 1, 2, \dots, N$ , to label the lattice sites in the quiver diagram. Note that for convenience the “background” in the diagram is drawn like a regular lattice, but remember that at this moment no notion of metric has been defined.

The orbifolded action, as a quiver mechanical model, reads

$$\begin{aligned}
S = \int d\tau \text{Tr} \{ & \frac{1}{2R_{11}} [D_\tau, Y^i]^2 + \frac{1}{2R_{11}} [D_\tau, Y^m]^2 \\
& + \frac{R_{11}}{4} [Y^i, Y^j]^2 + \frac{R_{11}}{2} [Y^i, Y^m]^2 + \frac{R_{11}}{4} [Y^m, Y^n]^2 \\
& - \frac{i}{2} \lambda^\dagger [D_\tau, \lambda] + \frac{R_{11}}{2} \lambda^\dagger \gamma^i [Y_i, \lambda] + \frac{R_{11}}{2} \lambda^\dagger \gamma^m [Y_m, \lambda] \} \quad (13)
\end{aligned}$$

where  $i, j$  run from 1 to 3,  $m, n$  from 4 to 9. Or equivalently in terms of the complex coordinates  $Z^a$ ,

$$S = \int d\tau \text{Tr} \{ \frac{1}{2R_{11}} [D_\tau, Y^i]^2 + \frac{R_{11}}{4} [Y^i, Y^j]^2$$

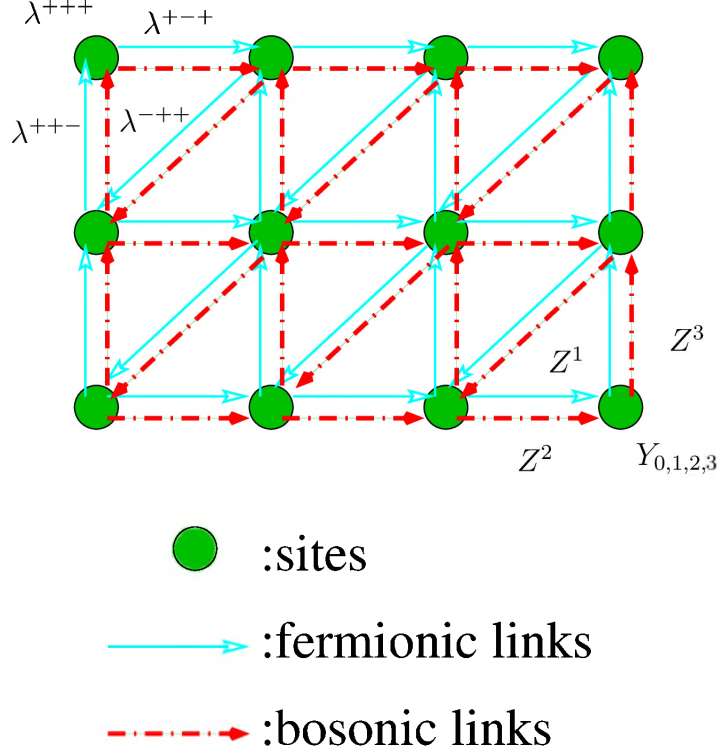


Figure 3: Quiver diagram for the Matrix Theory on  $\mathcal{C}^3/\mathcal{Z}_N^2$

$$\begin{aligned}
& + \frac{1}{R_{11}} [D_\tau, Z^a] [D_\tau, Z^{a\dagger}] + \frac{R_{11}}{2} ([Z^a, Z^{a'\dagger}] [Z^{a\dagger}, Z^{a'}] + [Z^a, Z^{a'}] [Z^{a\dagger}, Z^{a'\dagger}]) \\
& + R_{11} [Y^i, Z^a] [Y^i, Z^{a\dagger}] \\
& - \frac{i}{2} \lambda^\dagger [D_\tau, \lambda] + \frac{R_{11}}{2} \lambda^\dagger \gamma^i [Y_i, \lambda] + \frac{R_{11}}{\sqrt{2}} \lambda^\dagger (\tilde{\gamma}_a [Z^a, \lambda] + \tilde{\gamma}_a^\dagger [Z^{a\dagger}, \lambda]) \}. \quad (14)
\end{aligned}$$

In this action, the trace  $Tr$  is taken on a  $KN^2$ -by- $KN^2$  matrix, and  $\tilde{\gamma}_a = (\gamma^{2a+2} - i\gamma^{2a+3})/2$ , with

$$\begin{aligned}
\tilde{\gamma}^1 &= -\epsilon \otimes \tau_- \otimes \mathbf{1} \otimes \tau_3, \\
\tilde{\gamma}^2 &= i\mathbf{1} \otimes \tau_3 \otimes \tau_- \otimes \tau_3, \\
\tilde{\gamma}^3 &= -i\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \tau_-, \quad (15)
\end{aligned}$$

and  $\tau_- = (\tau_1 - i\tau_2)/2$ .

Recall that the number of surviving supersymmetries corresponds to the number



of fermionic variables on each site; so we have four supercharges corresponding to  $\lambda^{s_0+++}$  and  $\lambda^{s_0---}$  with  $(s_0 = 1, 2)$ . Relabel fermionic variables according their statuses in the quiver diagram,  $\Lambda_0^s = \lambda^{s+++}$ ,  $\Lambda_1^s = \lambda^{s-++}$ ,  $\Lambda_2^s = \lambda^{s+-+}$ ,  $\Lambda_3^s = \lambda^{s++-}$ . Hence, the Yukawa interactions in the action (14) can be recast into a tri-linear form:

$$S_{BF} = \int d\tau \sqrt{2} R_{11} \text{Tr} \{ -\Lambda_1^\dagger \epsilon [Z^1, \Lambda_0] + i\Lambda_2^\dagger [Z^2, \Lambda_0] - i\Lambda_3^\dagger [Z^3, \Lambda_0] \\ - [Z^1, \Lambda_2] \epsilon \Lambda_3 - i\Lambda_1 [Z^2, \Lambda_3] + i\Lambda_1 [\Lambda_2, Z^3] + h.c. \}. \quad (16)$$

As well-known in the literature on deconstruction, the allowed Yukawa interactions are such that a closed triangle (which may a degenerate one) is always formed by the two fermionic legs and one bosonic leg or site, due to the orbifolded gauge invariance.

## 4 Constructing $d = 2 + 1$ World-volume Theory

Previously in M(atric) Theory on a 2-torus, the compactified coordinates that solve the quotient conditions were expressed as covariant derivatives, resulting in a  $(2+1)$ -dimensional SYM on the dual torus with 16 supersymmetries. In this section, to provide a (de)construction of this  $d = 2 + 1$  SYM, in the same spirit as advocated in Ref. [14] and in Sec. 2, we will assign non-vanishing VEV to the bosonic link variables in our quiver mechanical model with the action (14), and show that it will approach in the continuum limit (i.e., a large  $N$  limit) to the desired SYM. Though the outcome may be expected in advance (see for example [19]), to see how the limit is achieved with a generic triangular lattice and how it could be reduced to a rectangular lattice should be helpful for our later discussion for wrapping matrix membranes.



## 4.1 Moduli and Parametrization of Fluctuations

To proceed, we first note that the orbifolding conditions (9) require the bi-fundamental bosonic matrix variables be of a particular block form in the site indices:  $(Z^a)_{mn,m'n'} = z^a(m, n)(\hat{V}_a)_{mn,m'n'}$ , with

$$\begin{aligned}(\hat{V}_2)_{mn,m'n'} &= (V_N)_{m,m'}(\mathbf{1}_N)_{n,n'}, \\(\hat{V}_3)_{mn,m'n'} &= (\mathbf{1}_N)_{m,m'}(V_N)_{n,n'},\end{aligned}\tag{17}$$

and  $\hat{V}_1 := \hat{V}_2^\dagger \hat{V}_3^\dagger$ . Each  $z^a(m, n)$  ( $a = 1, 2, 3$ ) (for fixed  $(m, n)$ ) is a  $K$ -by- $K$  matrix. (Since  $\hat{V}_2, \hat{V}_3$  can be viewed as the generators of the discrete group  $\mathcal{Z}_N \otimes \mathcal{Z}_N$ , the orbifolded matrices are of the form of the so-called *crossed product* of  $Mat(K)$  with  $\mathcal{Z}_N^2$  [24][25], expressing the properly projected form of the matrices upon orbifolding.)

In our approach, the key step after orbifolding is to assign nonzero VEV to each element  $z^a(m, n)$ :

$$z^a = \langle z^a \rangle + \tilde{z}^a,$$

with  $\langle z^a \rangle = c_a/\sqrt{2}$ ,  $c_a$  complex numbers to be specified later. Later in this paper we will take the following moduli:

$$c_1 = 0, c_2 = NR_2/2\pi, c_3 = NR_3/2\pi.\tag{18}$$

Accordingly, the fluctuations can be parameterized as

$$\begin{aligned}\tilde{z}^1 &= (\phi_1 + i\phi'_1)/\sqrt{2}, \\ \tilde{z}^2 &= (\phi_2 - iR_2A_2)/\sqrt{2}, \\ \tilde{z}^3 &= (\phi_3 - iR_3A_3)/\sqrt{2}.\end{aligned}\tag{19}$$

We remark that all the new variables appearing in the parametrization are  $K$ -by- $K$  matrices and that the parametrization (19) is intimately related with our choice (18) for the moduli.



The above decomposition has an obvious physical interpretation:  $\langle z^a \rangle$  are the modulus part and  $\tilde{z}^a$  the fluctuations. For the modulus part, as in Eq. (6) for our previous deconstruction of IIA string, we expect that the lattice constants are simply inversely proportional to the VEV of the bi-fundamental boson variable. Schematically  $a \propto 1/c_{2,3}$  such that  $aN \sim 1$ . Indeed we will see that the spatial geometry, after taking the continuum limit, emerges from three modulus parameters  $\langle z^a \rangle$  ( $a = 1, 2, 3$ ); in other words, a world-volume metric will be constructed out of them. Our choice (18) of moduli will greatly simplify the complexity that arises from our quiver diagram being a two dimensional triangular lattice. For the fluctuations, the parametrization in Eq. (19) corresponds to the infinitesimal form of the polar-coordinate decomposition of  $z^a$ . Note that the Yang-Mills fields come from the imaginary (or angular) part of the bi-fundamental fluctuations.

Before proceeding to consider the continuum limit, we make the following remarks on algebraic properties of the matrices  $\hat{V}_a$ . Let  $f$  be a diagonal matrix in the site indices:  $f_{mn,m'n'} = f(m, n)\delta_{mm'}\delta_{nn'}$ , then

$$\hat{V}_a^\dagger f \hat{V}_a = S_a f, \quad (20)$$

where  $S_a$  is the shift operator by a unit along  $a$ -direction, i.e.

$$S_1 f(m, n) = f(m - 1, n - 1), \quad (21)$$

$$S_2 f(m, n) = f(m + 1, n), \quad (22)$$

$$S_3 f(m, n) = f(m, n + 1). \quad (23)$$

Subsequently, one can easily derive the following relations for manipulations on a site function  $f$ :

$$\hat{V}_a f \hat{V}_a^\dagger = S_a^{-1} f, \quad \hat{V}_a (S_a f - f) = [f, \hat{V}_a], \quad [f, \hat{V}_a^\dagger] = -[f, \hat{V}_a]^\dagger. \quad (24)$$



## 4.2 World-volume Geometry from Deconstruction

In the following two subsections, we will examine the bosonic part of the action,

$$S_B = \int d\tau Tr \left\{ \frac{1}{R_{11}} |[D_\tau, Z^a]|^2 - \frac{R_{11}}{2} (|[Z^a, Z^{a\dagger}]|^2 + |[Z^a, Z^{a'}]|^2) \right. \\ \left. + \frac{1}{2R_{11}} [D_\tau, Y^i]^2 - R_{11} |[Y^i, Z^a]|^2 + \frac{R_{11}}{4} [Y^i, Y^j]^2 \right\}. \quad (25)$$

With the parametrization proposed above, the quadratic part for  $Y^i$  becomes

$$S_Y = \int d\tau Tr \left\{ \frac{1}{2R_{11}} [D_\tau, Y^i]^2 - R_{11} |[Y^i, Z^a]|^2 \right\} \\ = \int d\tau Tr \left\{ \frac{1}{2R_{11}} |\dot{Y}^i + i[Y_0, Y^i]|^2 - R_{11} |\partial_a Y^i \frac{2\pi S_a(z^a)}{N} + [Y^i, S_a(\tilde{z}^a)]|^2 \right\} \quad (26)$$

where we have introduced the lattice partial derivative,

$$\partial_a f := N(S_a f - f)/2\pi. \quad (27)$$

First we see the information of the metric on the would-be world-volume is contained in the term

$$S_{YK} = \int d\tau R_{11} Tr \left\{ \frac{1}{2R_{11}^2} [\dot{Y}^i]^2 - \frac{(2\pi)^2 |S_a(z^a)|^2}{N^2} [\partial_a Y^i]^2 \right\}. \quad (28)$$

Recall that the target space index  $i$  runs from 1 to 3 and that the index  $a$  also from 1 to 3. Note that  $\partial_1$  is not independent of  $\partial_{2,3}$ , because of the relation

$$\partial_1 = -S_2^{-1} S_3^{-1} \partial_2 - S_3^{-1} \partial_3.$$

Changing the unit for time  $\tau' := R_{11}\tau$  and then suppressing the prime for  $\tau'$ , Eq. (28) changes into

$$S_{YK} = \int d\tau \frac{1}{2} Tr \left\{ g^{00} [\dot{Y}^i]^2 - g^{22} [\partial_2 Y^i]^2 - g^{33} [S_2 \partial_3 Y^i]^2 - g^{23} \{ \partial_2 Y^i, S_2 \partial_3 Y^i \} \right\}, \quad (29)$$

from which we can read off the contravariant metric on world-volume

$$g^{00} = 1, g^{22} = \frac{8\pi^2 (|z^1|^2 + |S_2(z^2)|^2)}{N^2}, \\ g^{33} = \frac{8\pi^2 (|z^1|^2 + |S_1^{-1}(z^3)|^2)}{N^2}, g^{23} = \frac{8\pi^2 |z^1|^2}{N^2} \quad (30)$$



and  $g^{0k} = 0$  for the spatial index  $k = 2, 3$ . In this way, we see how the world-volume naturally metric arises from our (de)construction.

Now we are ready to take the continuum limit. First regularize the trace  $Tr$  to be  $\sum (2\pi)^2 tr / N^2 \kappa$ ; with our choice (18) of the moduli, we take  $\kappa = R_2 R_3$ . To see how world-volume geometry comes out of this construction, we consider the continuum limit with  $N \rightarrow \infty$ , then  $Tr \rightarrow \int d^2 \sigma tr / R_2 R_3$ , where  $tr$  is the trace over  $K$ -by- $K$  matrices. Here we have taken the spatial world-volume coordinates  $\sigma^{2,3}$  to run from 0 to  $2\pi$ . Subsequently, the action in the continuum limit reads

$$S_{YK} = - \int d\tau \int d^2 \sigma \frac{1}{R_2 R_3} tr \left\{ \frac{1}{2} g^{\alpha\beta} \partial_\alpha Y^i \partial_\beta Y^i \right\}, \quad (31)$$

in which the contravariant metric on  $(2+1)$ -dimensional world-volume is

$$(g^{\alpha\beta}) = \text{diag} (-1, R_2^2, R_3^2). \quad (32)$$

In fact, we have specified the behavior of fluctuations  $\tilde{z}^a$ , in the continuum limit  $N \rightarrow \infty$ , as  $\mathcal{O}(1)$ ; namely the contribution from the fluctuations to the world-volume metric in Eq. (30) smears or smoothes in the large- $N$  limit. As another fact, due to our choice  $c_1 = 0$ , world-volume metric (30) becomes diagonal, as shown in Eq. (32). Accordingly, the factor  $\kappa$ , taken to be  $R_2 R_3$ , multiplied by the coordinate measure  $d^2 \sigma$ , is nothing but the invariant world volume measure, namely  $\kappa = \sqrt{-\det(g_{\alpha\beta})}$ , where

$$(g_{\alpha\beta}) = \text{diag} (-1, 1/R_2^2, 1/R_3^2) \quad (33)$$

A further check shows that the area of the world-volume torus is

$$\begin{aligned} & \text{Area of world-volume torus} \\ &= \sqrt{-\det(g_{\alpha\beta})} \cdot \text{coordinate area} = \frac{(2\pi)^2}{R_2 R_3}. \end{aligned} \quad (34)$$

Now for the continuum limit of the full  $S_Y$ , it is easy to show that

$$S_Y = \int d\tau d^2 \sigma \frac{1}{2 R_2 R_3} tr \left\{ -g^{\alpha\beta} D_\alpha Y^i D_\beta Y^i + [\phi'_1, Y^i]^2 + [\phi_a, Y^i]^2 \right\} \quad (35)$$



where  $D_\alpha = \partial_\alpha + i[A_\alpha, \cdot]$  and  $A_0 = Y^0$ .

In this subsection we have constructed, from our quiver mechanics, the toroidal geometry on the spatial world-volume, as well as the standard gauge interactions of world-volume scalar fields. (Incidentally let us observe that besides the local world-volume geometry, the modular parameter that describes the shape of the compactified 2-torus in target space is also dictated by the moduli  $\langle z^a \rangle$  that also fix the shape of the triangular unit cell in the quiver diagram. However, in this paper we will concentrate on our special choice (18) for the moduli, and leave the analysis of more general moduli to the sequential paper.)

### 4.3 Yang-Mills Field Construction

Now we consider the action of bi-fundamental fields; our goal here is to construct the desired SYM.

Collecting the relevant terms in the action, we have

$$S_Z = \int d\tau Tr \left\{ \frac{1}{R_{11}} |[D_\tau, Z^a]|^2 - \frac{R_{11}}{2} (|[Z^a, Z^{a'\dagger}]|^2 + |[Z^a, Z^{a'}]|^2) \right\}. \quad (36)$$

Write  $[Z^{a'\dagger}, Z^a] = \hat{V}_{a'}^\dagger P_{a'a} \hat{V}_a$ , with  $P_{a'a} = z^{a'\dagger} z^a - S_{a'}^{-1} z^a S_a^{-1} z^{a'\dagger}$ . Separating the VEV and fluctuations,

$$P_{a'a} = S_{a'}^{-1} \partial_{a'} \tilde{z}^a \frac{2^{-1/2} \bar{c}_{a'} + S_a^{-1} \tilde{z}^{a'\dagger}}{(2\pi)^{-1} N} + \frac{2^{-1/2} c_a + \tilde{z}^a}{(2\pi)^{-1} N} S_a^{-1} \partial_a \tilde{z}^{a'\dagger} + [\tilde{z}^{a'\dagger}, \tilde{z}^a].$$

Therefore,  $P_{a'a}^\dagger = P_{aa'}$ . Moreover,  $[Z^a, Z^{a'}] = Q_{a'a} \hat{V}_a \hat{V}_{a'}$ , with  $Q_{a'a} = z^a S_a^{-1} z^{a'} - z^{a'} S_{a'}^{-1} z^a$ ; thus

$$Q_{a'a} = S_{a'}^{-1} \partial_{a'} \tilde{z}^a \frac{2^{-1/2} c_{a'} + S_a^{-1} \tilde{z}^{a'}}{(2\pi)^{-1} N} - \frac{2^{-1/2} c_a + S_{a'}^{-1} \tilde{z}^a}{(2\pi)^{-1} N} S_a^{-1} \partial_a \tilde{z}^{a'} + [S_{a'}^{-1} \tilde{z}^a, \tilde{z}^{a'}].$$

Hence  $Q_{a'a} = -Q_{aa'}$ . The action (36) is recast into

$$\begin{aligned} S_Z &= \int d\tau Tr \left\{ \frac{1}{R_{11}} |[D_\tau, Z^a]|^2 - \frac{R_{11}}{2} \left( \sum_a P_{aa}^2 + 2 \sum_{a' < a} (|P_{a'a}|^2 + |Q_{a'a}|^2) \right) \right\} \\ &= \int d\tau Tr \left\{ |S_{a0}|^2 - \left( \frac{1}{2} \sum_a P_{aa}^2 + \sum_{a' < a} (|P_{a'a}|^2 + |Q_{a'a}|^2) \right) \right\} \end{aligned} \quad (37)$$



where, in the second line, a rescaling of time is implied, and  $S_{a0} = 2\pi S_a^{-1} \partial_a A_0 z^a / N - i\tilde{z}^a + [S_a^{-1} A_0, \tilde{z}^a]$ . Now we assign the VEV (18), and calculate the following intermediate quantities:

$$S_{10} = (-iD_0\phi_1 - D_0\phi'_1)/\sqrt{2}, \quad (38)$$

$$S_{k0} = (R_k F_{k0} - iD_0\phi_k)/\sqrt{2}, \quad (39)$$

$$P_{11} = i[\phi_1, \phi'_1], \quad (40)$$

$$P_{kk} = R_k D_k \phi_k, \quad (41)$$

$$P_{1k} = (R_k D_k \phi_1 - iR_k D_k \phi'_1 + [\phi_1, \phi_k] - i[\phi'_1, \phi_k])/2, \quad (42)$$

$$Q_{1k} = (-R_k D_k \phi_1 - iR_k D_k \phi'_1 - [\phi_1, \phi_k] - i[\phi'_1, \phi_k])/2, \quad (43)$$

$$P_{23} = (R_2 D_2 \phi_3 + R_3 D_3 \phi_2 + [\phi_2, \phi_3] - iR_2 R_3 F_{23})/2, \quad (44)$$

$$Q_{23} = (R_2 D_2 \phi_3 - R_3 D_3 \phi_2 - [\phi_2, \phi_3] - iR_2 R_3 F_{23})/2 \quad (45)$$

and their squares

$$|S_{10}|^2 = (|D_0\phi_1|^2 + |D_0\phi'_1|^2)/2, \quad (46)$$

$$|S_{k0}|^2 = (R_k^2 |F_{k0}|^2 + |D_0\phi_k|^2)/2, \quad (47)$$

$$|P_{1k}|^2 = (|R_k D_k \phi_1 - i[\phi'_1, \phi_k]|^2 + |R_k D_k \phi'_1 + i[\phi_1, \phi_k]|^2)/4, \quad (48)$$

$$|Q_{1k}|^2 = (|R_k D_k \phi_1 + i[\phi'_1, \phi_k]|^2 + |R_k D_k \phi'_1 - i[\phi_1, \phi_k]|^2)/4, \quad (49)$$

$$|P_{23}|^2 = (|R_2 D_2 \phi_3 + R_3 D_3 \phi_2|^2 + |[\phi_2, \phi_3] - iR_2 R_3 F_{23}|^2)/4, \quad (50)$$

$$|Q_{23}|^2 = (|R_2 D_2 \phi_3 - R_3 D_3 \phi_2|^2 + |[\phi_2, \phi_3] + iR_2 R_3 F_{23}|^2)/4. \quad (51)$$

Here the gauge field strength  $F_{\alpha\beta}$  is defined in the usual way. Subsequently, we have, in the continuum limit,

$$S_Z = \int d\tau d^2\sigma \frac{1}{R_2 R_3} \text{tr} \left\{ -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} D_\alpha \phi^M D_\beta \phi^M + \frac{1}{4} [\phi^M, \phi^N]^2 \right\}, \quad (52)$$

in which  $\phi^M$  include  $\phi_{1,2,3}$  and  $\phi'_1$ .

In summary, Eqs. (35) and (52) leads to the complete continuum bosonic action:

$$S_B = \int d\tau d^2\sigma \frac{1}{R_2 R_3} \text{tr} \left\{ -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} D_\alpha \Phi^I D_\beta \Phi^I + \frac{1}{4} [\Phi^I, \Phi^I]^2 \right\} \quad (53)$$

where  $\Phi^I$  include both  $\phi^M$  and  $Y^i$ .

Eq. (53) is the dimensional reduction from the bosonic part of either four-dimensional  $\mathcal{N} = 4$  SYM or ten-dimensional  $\mathcal{N} = 1$  SYM. Therefore, with appropriate rescaling of world-volume fields, the action (53) possesses an  $O(7)$  symmetry for fields  $Y^i$ ,  $\phi_a$  and  $\phi'_1$ .

We remark that the first two terms in the continuum action (53) contain one and the same world-volume metric. This can not be a mere coincidence; it is dictated by the stringent internal consistency of our deconstruction procedure.

#### 4.4 Fermion Construction

The deduction of the continuum action for fermionic variables can be carried out in a similar way. We only emphasize that the phase of D-particle moduli can be absorbed into redefinition of fermionic variables in Eq. (16). Indeed after assigning non-zero VEV (or moving the D-particles collectively away from the orbifold singularity), the free fermionic action becomes

$$\begin{aligned}
 S_{FF} = & \int d\tau \frac{(2\pi)^2}{N^2 \kappa} \sum_{m,n} \text{tr} \{ \lambda_{m,n}^{1\dagger} (-) \epsilon c_1 (\lambda_{m+1,n+1}^0 - \lambda_{m,n}^0) \\
 & + \lambda_{m,n}^{2\dagger} i c_2 (\lambda_{m-1,n}^0 - \lambda_{m,n}^0) + \lambda_{m,n}^{3\dagger} (-i) c_3 (\lambda_{m,n-1}^0 - \lambda_{m,n}^0) \\
 & - (\lambda_{m+1,n}^2 - \lambda_{m,n-1}^2) \epsilon c_1 \lambda_{m,n}^3 + i \lambda_{m,n}^1 c_2 (\lambda_{m+1,n+1}^3 - \lambda_{m,n+1}^3) \\
 & + i \lambda_{m,n}^1 c_3 (\lambda_{m+1,n+1}^3 - \lambda_{m+1,n}^3) + h.c. \}
 \end{aligned} \tag{54}$$

where we have introduced  $\lambda_a$  by writing  $(\Lambda_a)_{mn,m'n'} = (\lambda_a)_{mn} (\hat{V}_a)_{mn,m'n'}$  for  $a = 0, 1, 2, 3$ , with  $\hat{V}_0$  the unit matrix. With the choice (18) and  $\kappa = R_2 R_3$ , the continuum limit reads

$$\begin{aligned}
 S_{FF} = & \int d\tau d^2\sigma \frac{1}{R_2 R_3} \text{tr} \{ \lambda_2^\dagger (-i) R_2 \partial_2 \lambda_0 + \lambda_3^\dagger i R_3 \partial_3 \lambda_0 \\
 & \lambda_1 i R_2 \partial_2 \lambda_3 + \lambda_1 i R_3 \partial_3 \lambda_2 + h.c. \}.
 \end{aligned} \tag{55}$$

Comparing with our definition of gamma matrices, (10) and (15), we get

$$S_{FF} = \int d\tau d^2\sigma \frac{1}{R_2 R_3} \text{tr} \{ \lambda^\dagger i (\gamma^7 R_2 \partial_2 + \gamma^9 R_3 \partial_3) \lambda \} / 2. \quad (56)$$

The appearance of the seventh and ninth gamma matrices can be easily understood from the parametrization of the fluctuations in Eq. (19), since the gauge fields arise from the fluctuations in the imaginary part of complex coordinates  $\tilde{z}^a$ .

The continuum limit of the complete action for fermionic variables is thus

$$S_F = \int d\tau d^2\sigma \frac{1}{R_2 R_3} \text{tr} \left\{ -\frac{i}{2} \lambda^\dagger [D_\tau, \lambda] + \frac{i}{2} \lambda^\dagger \gamma^7 R_2 [D_2, \lambda] + \frac{i}{2} \lambda^\dagger \gamma^9 R_3 [D_3, \lambda] + \frac{1}{2} \lambda^\dagger \gamma^I [\Phi^I, \lambda] \right\}. \quad (57)$$

Summarily, Eq. (53) plus Eq. (57) constitute precisely the desired  $d = 2+1$  SYM with sixteen supercharges. This outcome in the continuum limit verifies our idea that M(atrrix) Theory compactification can be deconstructed in terms of orbifolding and quiver mechanics, so that we are on the right track to IIB/M duality. Moreover, the constructed world-volume geometry, in view of world-volume/target-space duality, provides the platform for the discussion in next section on the matrix membranes wrapping on the compactified 2-torus in target space.

## 5 Wrapping Membranes and Their Spectrum

In the above we have “(de)constructed” the M(atrrix) Theory compactified on a 2-torus, by starting from orbifolding and then assigning non-zero VEVs to bi-fundamental bosons. According to the IIB/M duality conjecture, M(atrrix) Theory compactified on a 2-torus is dual to IIB string theory on a circle. In this section, we will show that the quiver mechanics resulting from our deconstruction procedure indeed possesses features that are characteristic to the IIB/M duality. More concretely, we will explicitly construct matrix states in our quiver mechanics, which in the continuum limit correspond to membranes wrapping over compactified a 2-torus

in the target space. And we will show a  $SL(2, \mathcal{Z})$  symmetry for these states and the emergence from these states of a new flat dimension when the compactified 2-torus shrinks to zero. As mentioned in Sec. 1, these features are central for verifying the IIB/M duality conjecture. The existence of these states and their properties were discussed in the literature of M(atric) Theory in the context of  $d = 2 + 1$  SYM, but to our knowledge the explicit matrix construction of these states is still absent in the literature.

## 5.1 Classical Wrapping Membrane

To motivate our construction, we recall that a closed string winding around a circle is described by a continuous map from the worldsheet circle to the target space circle, satisfying the periodic condition:  $x(\sigma + 2\pi) = x(\sigma) + 2\pi w R$ , where  $w$  is the winding number. For a  $\mathcal{C}^2/\mathcal{Z}_N$  orbifold, a winding string is a state in the twisted sector satisfying  $z(\sigma + 2\pi) = e^{i2\pi w/N} z(\sigma)$ .

In the same way, in the non-linear sigma model for a toroidal membrane wrapping on a compactified 2-torus, we have the boundary conditions

$$\begin{aligned} x_1(p + 2\pi, q) &= x_1(p, q) + 2\pi m R_1, & x_1(p, q + 2\pi) &= x_1(p, q) + 2\pi n R_1, \\ x_2(p + 2\pi, q) &= x_2(p, q) + 2\pi s R_2, & x_2(p, q + 2\pi) &= x_2(p, q) + 2\pi t R_2 \end{aligned} \quad (58)$$

where  $(p, q)$  ( $0 \leq p, q \leq 2\pi$ ) parameterize the membrane. The solutions to these conditions are given in Eq. (1). They contain four integers  $(m, n, s, t)$ . One combination of them is  $SL(2, \mathcal{Z})$  invariant, i.e. the *winding/wrapping number*:

$$w = mt - ns. \quad (59)$$

(This is a slight modification of Schwarz's original construction [2][26]; note that this combination appeared also in the context of noncommutative torus, see for example Eq. (4.2) in [27]). The geometric significance of  $w$  is simply the ratio of the pullback



area of the membrane to the area of the target torus. In IIB/M duality,  $w$  is related to the Kaluza-Klein momentum in the newly emergent dimension.

For a membrane wrapping on the orbifold  $\mathcal{C}^3/\mathcal{Z}_N^2$ , the twisted boundary conditions become

$$\begin{aligned} z^2(p + 2\pi, q) &= e^{i2\pi m/N} z^2(p, q) \quad , \quad z^2(p, q + 2\pi) = e^{i2\pi n/N} z^2(p, q), \\ z^3(p + 2\pi, q) &= e^{i2\pi s/N} z^3(p, q) \quad , \quad z^3(p, q + 2\pi) = e^{i2\pi t/N} z^3(p, q), \end{aligned} \quad (60)$$

as well as

$$z^1(p + 2\pi, q) = e^{-i2\pi(m+s)/N} z^1(p, q), \quad z^1(p, q + 2\pi) = e^{i2\pi(n+t)/N} z^1(p, q). \quad (61)$$

Their solutions are of the form

$$\begin{aligned} z^1(p, q; \tau) &= 2^{-1/2} c_1(\tau) e^{-i[(m+s)p + (n+t)q]/N} \cdot (\text{oscillator modes}), \\ z^2(p, q; \tau) &= 2^{-1/2} c_2(\tau) e^{i(mp+nq)/N} \cdot (\text{oscillator modes}), \\ z^3(p, q; \tau) &= 2^{-1/2} c_3(\tau) e^{i(sp+ tq)/N} \cdot (\text{oscillator modes}). \end{aligned} \quad (62)$$

Here the coefficients  $c_a(\tau)$  describe the center-of-mass degrees of freedom. Again, the solutions contain a quadruple,  $(m, n, s, t)$ , of integers, giving rise to an  $SL(2, \mathcal{Z})$  invariant wrapping number  $w = mt - ns$ .

Now let us consider a stretched membrane (62), with the oscillator modes suppressed (by setting the corresponding factor to unity). The membrane stretching energy density is known to be proportional to  $\{Y^I, Y^J\}\{Y^I, Y^J\}$ , in terms of Poisson bracket with respect to the canonical symplectic structure on the membrane spatial world-volume. (See, e.g., Ref. [28] for a latest review and also for a comprehensive list of references.) It is easy to show that with choosing  $c_1 = 0$ ,  $c_{2,3} \propto NR_{2,3}$ , the Poisson bracket of  $z^2(p, q)$  and  $z^3(p, q)$  is

$$\{z^2(p, q), z^3(p, q)\} \propto w R_2 R_3. \quad (63)$$

So the energy of this wrapping configuration is proportional to  $(wR_2R_3)^2$ , having the correct dependence on  $w$  and on the area of the 2-torus in target space, because the choice of  $c_a$  corresponds to a regular target torus.

We note that to show IIB/M duality in the context of M(atrix) Theory, we need a construction of wrapping membranes either in the setting of  $d = 2 + 1$  SYM or in that of our quiver mechanics. But the above construction (62) is in neither, so it is *not* what we are looking for. However, we may view it as a sort of the classical and continuum limit of the wrapping matrix membrane we are looking for, pointing to the direction we should proceed.

## 5.2 Wrapping Matrix Membranes

Now we try to transplant the above continuum scenario into the notion of matrix membrane. It is well-known that for the finite matrix regularization of a membrane, whose topology is a torus, one needs to prompt the membrane coordinates into the clock and shift matrices, namely to substitute  $e^{ip} \rightarrow U_K$  and  $e^{iq} \rightarrow V_K$  [28], where  $K$  is the number of D-particles that constitute the membrane. The key property of  $U_K$  and  $V_K$  for this substitution to work is

$$U_K^m V_K^n = (\omega_K)^{mn} V_K^n U_K^m, \quad (64)$$

with arbitrary integers  $m, n$ , and  $\omega_K = e^{i2\pi/K}$ . In the classical or continuum limit, it gives to the correct Poisson bracket between  $p$  and  $q$ . Eq. (62) motivates us to deal with the fractional powers of the clock and shift matrices. The guide line to formulate the arithmetics of the fractional powers is to generalize Eq. (64) to be valid for fractional powers. Below we model a simplest substitution scheme

$$e^{ip/N} \rightarrow U_{KN^2} =: U_K^{1/N}, e^{iq/N} \rightarrow V_{KN^2} =: V_K^{1/N}, \quad (65)$$

which is enough at the current stage to enable us to explore the physics of IIB/M duality. In accordance with these substitutions, we introduce the following *Ansatz*



for the matrix configuration of a wrapping membrane:

$$\begin{aligned} Z_{(mem)}^1 &= 2^{-1/2} c_1 U^{-(m+s)} V^{-(n+t)} \otimes \hat{V}_1, \\ Z_{(mem)}^2 &= 2^{-1/2} c_2 U^m V^n \otimes \hat{V}_2, \\ Z_{(mem)}^3 &= 2^{-1/2} c_3 U^s V^t \otimes \hat{V}_3. \end{aligned} \quad (66)$$

with omitted subscripts  $U := U_{KN^2}, V := V_{KN^2}$ .

These matrices are  $KN^4$ -by- $KN^4$ , of the form of a direct product of three factors, with the rank  $KN^2$ ,  $N$  and  $N$  respectively. In our quiver mechanics, the first factor describes  $K$  D-particles on the  $Z_N^2$  orbifold while the last two factors describe the deconstructed torus coordinates. The form of the last two factors in *Ansatz* (66), say  $\hat{V}_2$  in  $Z_{(mem)}^2$ , is dictated by the orbifolding conditions. The form of the first factor, say  $U^m V^n$  in  $Z_{(mem)}^2$  is motivated by Eqs. (62). In other words, the expression  $c_2 U^m V^n / \sqrt{2}$  can be viewed as polar coordinate decomposition of  $Z_{(mem)}^2$ , in which  $c_2$  is interpreted as the distance of the membrane on the orbifold to the singularity, while the unitary matrix  $U^m V^n$  shows how the membrane constituents wrap in the angular direction of  $Z^2$ . A similar interpretation holds for  $Z_{(mem)}^3$  and  $Z_{(mem)}^1$ .

Though the above intuitive picture looks satisfactory, the real justification for the *Ansatz* (66) to describe wrapping membranes on a compactified 2-torus should come the detailed study of the physical properties of these states in accordance with M(atric) Theory, which will be carried out in next a few subsections.

### 5.3 Spectrum and $SL(2, \mathcal{Z})$ Invariance

From now on, we will suppress the subscript in  $Z_{(mem)}^a$ . First let us verify that the equations of motion, that follow from the action (14),

$$\ddot{Z}^a + \frac{R_{11}^2}{2} ([Z^b, [Z^{b\dagger}, Z^a]] + [Z^{b\dagger}, [Z^b, Z^a]]) = 0. \quad (67)$$

can be satisfied by the *Ansatz* (66). Indeed, from the experience of the equation of motion for the spherical matrix membrane, to solve the time-dependence of the

scalar coefficients ( $c_a$  here) is by far kind of involved [29][30]; for the toroidal case, however, this situation is simplified because the coefficient functions are complex. It is a straightforward calculation to show that the equations of motion (67) are satisfied if we take  $c_a(\tau) = c_{a0}e^{-i\omega_a\tau}$ , with

$$\omega_a^2 = R_{11}^2(1 - \cos \frac{2\pi w}{KN^2}) \sum_{b \neq a} |c_{b0}|^2. \quad (68)$$

Consistent with our previous choice (18) for the moduli, hereafter we will just take  $c_{10} = 0$ ,  $c_{k0} = NR_k/2\pi$  for ( $k = 2, 3$ ). Accordingly, we have

$$\omega_2^2 = ((2\pi)^{-1}NR_{11}R_3)^2(1 - \cos \frac{2\pi w}{KN^2}), \quad (69)$$

$$\omega_3^2 = ((2\pi)^{-1}NR_{11}R_2)^2(1 - \cos \frac{2\pi w}{KN^2}). \quad (70)$$

Moreover concerning the membrane dynamics, one can easily check an additional constraint equation

$$[\dot{Z}^a, Z^{a\dagger}] + [\dot{Z}^{a\dagger}, Z^a] = 0, \quad (71)$$

which plays the role of the gauge fixing of the membrane world-volume diffeomorphism symmetry; see, e.g., ref. [28].

Next let us examine the energy of the wrapping membrane states (66). By a direct calculation, we find that the potential (or stretching) energy density (*before* taking the trace  $Tr$ ) is proportional to the *unit* matrix,  $\mathbf{1}_{N^4}$ :

$$V = R_{11} \sum_{a < b} \{ |[Z^a, Z^{b\dagger}]|^2 + |[Z^a, Z^b]|^2 \} = (N/2\pi)^4 R_{11} (1 - \cos \frac{2\pi w}{KN^2}) (R_2 R_3)^2 \mathbf{1}_{N^4}, \quad (72)$$

implying a uniform stretching energy density on the membrane. We regularize the trace  $Tr$  to be  $(N^4)^{-1}(2\pi)^2 \sum$ ; therefore  $Tr \mathbf{1}_{KN^4} = (2\pi)^2 K$ . Then the total stretching energy is

$$E_w := Tr V = (2\pi)^{-2} K N^4 R_{11} (1 - \cos \frac{2\pi w}{KN^2}) (R_2 R_3)^2. \quad (73)$$



Including the kinetic energy due to the  $\tau$ -dependence of  $c_a$ , the total energy of the membrane is given by

$$\begin{aligned} H_{mem} &= Tr\left\{\frac{1}{R_{11}} \sum_{a=1}^3 |\dot{Z}^a|^2 + V\right\} \\ &= \frac{1}{2R_{11}} \sum_{a=2}^3 |\dot{c}_a|^2 Tr \mathbf{1}_{N^4} + E_w \\ &= 2E_w. \end{aligned} \quad (74)$$

A characteristic feature of Eqs. (73) and (74) is that though the states given by *Ansatz* (66) contain four integers  $(m, n; s, t)$ , their energy only depends on the wrapping number  $w = mt - ns$ , which is known invariant under the  $SL(2, \mathcal{Z})$  transformations

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}, \quad \begin{pmatrix} s' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}, \quad (75)$$

where  $a, b, c, d$  are integers, satisfying  $ad - bc = 1$ . This result is one chief evidence to justify our matrix construction (66), showing the  $SL(2, \mathcal{Z})$  symmetry of the spectrum of the wrapping membranes directly and explicitly in a constructive manner; this  $SL(2, \mathcal{Z})$  in Eqs. (75) is right the modular symmetry of the target torus, as well being interpreted as the S-duality in IIB string theory.

## 5.4 Emergent Flat Dimension

In last subsection, we have seen that the stretched energy  $E_w$  in Eq. (73) contains a factor  $(1 - \cos(2\pi w / KN^2))$ . Such factor is a common praxis in the lattice (gauge) field theory for the spectrum of the lattice Laplacian. In the large- $N$  limit, when  $w$  is small compared to  $N^2$ , one has  $(1 - \cos(2\pi w / KN^2)) \rightarrow 2\pi^2 w^2 / K^2 N^4$ . Hence  $E_w$  recovers the  $w^2(R_2 R_3)^2$  behavior of the Poisson bracket calculation for continuous membranes in previous subsection 5.1. (See Eq.(63) and discussion below it). More precisely, we have

$$E_w = w^2 R_{11} (R_2 R_3)^2 / 2K. \quad (76)$$

Consequently, it is amusing to note that this stretched energy can be rewritten as the light-cone energy due to a transverse momentum  $p_w$ :

$$E_w = \frac{p_w^2}{2P^+}, \quad p_w = w \cdot R_2 R_3, \quad (77)$$

in which the light-cone mass  $P^+ = K/R_{11}$ ! This energy is of the form of the energy for an excitation with Kaluza-Klein momentum  $w$  on a transverse circle with the radius  $1/R_2 R_3$ . (After recovering the Planck length, this radius is  $l_p^3/R_2 R_3$ ).

We assert that the spectrum of wrapping membranes is identical to that of excitations on a transverse circle. In fact, though there exists the apparent degeneration due to the states with the same wrapping number  $w$  for different combinations of  $(m, n, s, t)$ , however, as noted by Schwarz [26], this degeneration can be easily eliminated by considering F-D string spectrum and  $SL(2, \mathcal{Z})$  equivalence. Therefore, with wrapping membrane states viewed as momentum states, a new flat dimension opens up when the compactified 2-torus, on which membranes wrap, shrinks to zero (i.e.  $R_2 R_3 \rightarrow 0$ ). As mentioned in the introduction, this is exactly what the IIB/M duality requires for a M-theory formulation of IIB string theory.

## 5.5 Membrane Tension

To verify that the membrane has the correct tension, we note that the light cone energy in M theory for a stretched transverse membrane is given by

$$E_m = \frac{M_w^2}{2P^+}, \quad (78)$$

in which  $M_w = w T_{mem} A_{T^2}$ .  $T_{mem} = 1/(2\pi)^2$  is the membrane tension, with Planck length taken to be unity.  $A_{T^2}$  is the area of the compactified 2-torus in target space, which in our case is regular with sides  $2\pi R_2$  and  $2\pi R_3$ ; so  $A_{T^2} = R_2 R_3 (2\pi)^2$ . Therefore we have

$$E_m = w^2 R_{11} (R_2 R_3)^2 / 2K. \quad (79)$$

So in the large- $N$  limit, we have the equality:

$$E_w = E_m. \quad (80)$$

This is a perfect and marvelous match! An equivalent statement is that our wrapping matrix membrane indeed has the correct tension as required by M-theory.

An astute reader may have noticed that the energy (74) of our membrane configuration contains a kinetic part, which also equals  $E_w$ . We are going to clarify its origin in the next section.

## 5.6 Adding center-of-mass momentum

To better understand the origin of the kinetic part in the energy (74), which is equal to the stretching energy  $E_w$ , we try to add a generic center-of-mass momentum to the membrane states (66).

Let us first examine a closed string. A winding state on a compactified circle satisfies  $x_w(\sigma + 2\pi, \tau) = x_w(\sigma, \tau) + 2\pi w R$ , with  $R$  the radius of the circle. The periodic boundary condition is a linear homogenous relation, therefore the momentum quantum number can be added as a zero-mode part  $x_0$ , which is linear in  $\tau$ , such that  $x_w + x_0$  still satisfies the same equation.

Now we turn to the orbifold description at large  $N$  of the same fact, in which  $x/R$  becomes the angular part of a complex coordinate  $z$  and the radius of this circle is  $NR$ . At large  $N$  we can make the following substitution:

$$z = NR + i(x_w + x_0) \leftrightarrow NR e^{i x_w / NR} + i x_0. \quad (81)$$

The equation of motion for the string on the  $\mathcal{C}/\mathcal{Z}_N$  orbifold gives the solution  $x_w/R = w(\sigma - \tau)$ , for  $w > 0$ ,  $x_w/R = w(\sigma + \tau)$ , for  $w < 0$  and  $x_0 = p\tau$ . Suppose  $w > 0$  for definiteness. One finds the energy density is given by

$$\begin{aligned} h &= |\dot{z}|^2 + |z'|^2 \\ &= (p^2 + (Rw)^2 - 2pRw \cos \frac{w}{N}(\sigma - \tau)) + (Rw)^2. \end{aligned} \quad (82)$$

It is amusing to observe that in the large- $N$  limit,  $h$  approaches to  $(p - Rw)^2 + (Rw)^2$ . With  $p = 0$ , the kinetic energy is equal to the stretching energy! This feature is the same as we have encountered in Eq. (74), indicating that this is a common phenomenon in approaching compactification via orbifolding.

Giving a bit more details, we note that the orbifolding solution, in the large- $N$  limit, becomes  $z \rightarrow NR + i(p - Rw)\tau + iRw\sigma$ . Accordingly, we should interpret the linear combination  $p - Rw$  as the momentum in the continuum limit, though naively one might have expected that  $p$  would be identified as the center-of-mass momentum. The same analysis can be applied to the continuous membrane wrapping on an orbifold:

$$\begin{aligned} z^2(p, q; \tau) &= 2^{-1/2}(ik_2\tau + NR_2e^{i(N^{-1}(mp+nq)-\omega_2\tau)}), \\ z^3(p, q; \tau) &= 2^{-1/2}(ik_3\tau + NR_3e^{i(N^{-1}(sp+ tq)-\omega_3\tau)}). \end{aligned} \quad (83)$$

A similar analysis shows again that the momenta in the continuum limit are shifted by the wrapping number:  $p_a = k_a - NR_a\omega_a$ , for  $a = 2, 3$ . Thus with  $k_a = 0$ , a membrane wrapping on a torus is not at rest!

The states in Eqs. (81) and (83) can be generalized to matrix membranes, with center-of-mass momentum added:

$$\begin{aligned} Z^2 &= 2^{-1/2}(iR_{11}k_2\tau/2\pi K + c_2(\tau)U^mV^n) \otimes \hat{V}_2, \\ Z^3 &= 2^{-1/2}(iR_{11}k_3\tau/2\pi K + c_3(\tau)U^sV^t) \otimes \hat{V}_3 \end{aligned} \quad (84)$$

where, for a regular torus in target space, we have set  $Z^1 = 0$ . As in the previous section, we take  $c_a = c_{a0}e^{-i\omega_a\tau}$  with  $\omega_a$  ( $a = 2, 3$ ) in Eq. (69). In large- $N$  limit, the matrix membrane configuration in Eq. (84) is expected to be equivalent to the continuous configuration in Eq. (83) up to a normalization.

The kinetic energy density in the discrete setting,  $T := R_{11}^{-1}|\dot{Z}_a|^2$ , can be easily computed:

$$T = \frac{1}{2R_{11}}[(\frac{R_{11}k_a}{2\pi K})^2 + \omega_a^2|c_a|^2 - ((\frac{R_{11}k_a}{2\pi K})\omega_a c_a U^{m_a}V^{n_a} + h.c.)] \otimes \mathbf{1}_{N^2}, \quad (85)$$



in which the summation is over  $a = 2, 3$ ,  $m_2 = m$ ,  $n_2 = n$ ,  $m_3 = s$ ,  $n_3 = t$ . Adding the  $T$  in (85) to the  $V$  in Eq. (72), we get the total energy density, which results in the total energy of a wrapping membrane configuration with momenta and wrapping:

$$E_{k_2, k_3; w} = \frac{R_{11}}{2K} \left( k_a - \frac{2\pi K}{R_{11}} \omega_a |c_a| \right)^2 + E_w. \quad (86)$$

We are glad, just for convenience, to rewrite

$$\omega_2 = (\sqrt{2}\pi)^{-1} N R_{11} R_3 \sin(\pi w / K N^2), \quad (87)$$

$$\omega_3 = (\sqrt{2}\pi)^{-1} N R_{11} R_2 \sin(\pi w / K N^2), \quad (88)$$

$$|c_a| = N R_a / 2\pi. \quad (89)$$

Accordingly, the genuine momentum is identified to be

$$p_a = k_a - (\sqrt{2}\pi)^{-1} K N^2 R_2 R_3 \sin(\pi w / K N^2) \rightarrow k_a - R_2 R_3 w / \sqrt{2}. \quad (90)$$

In closed string theory, the quantization of momentum is a consequence of the single-valuedness of the translation operator  $e^{i2\pi R_a \hat{p}_a}$  (no summation on  $a$ ). In our present case, we can also impose the quantization condition at large  $N$ ; as a result  $p_a = l_a / R_a$  where  $l_a$  is an integer (or equivalently, one has  $k_a = l_a / R_a + R_2 R_3 w / \sqrt{2}$ ). Note that the domain of  $(p_2, p_3)$  is just the lattice dual to the target torus; therefore, the above equation is once more a manifestation of the covariance for  $E_{k_2, k_3; w}$  under the  $SL(2, \mathcal{Z})$  transformations over the target torus.

In summary, our explicit construction of wrapping membrane states, the appearance of  $SL(2, \mathcal{Z})$  symmetry in their spectrum, and an emergent flat dimension as well, all these combined together, constitute strong evidence for our quiver matrix mechanical model to be a non-perturbative formulation of IIB string theory, which naturally exhibits IIB/M duality.

## 6 Discussions and Perspectives

Problems that remain for future study and some perspectives are collected in this section. Some of them are just slightly touched in this work. (As mentioned before, we will leave to a sequential paper the discussions on generic moduli for the VEV of  $Z^a$  ( $a = 1, 2, 3$ ), which would allow both the SYM world-volume torus and the compactified target space torus non-regular, and would explicitly demonstrate the  $SL(2, \mathcal{Z})$  duality of our approach to IIB string theory.)

1. There have been three different ideas to deal with IIB/M duality, i.e. M-theory on a 2-torus should be dual to IIB string theory on a circle: namely the wrapping membrane states, the vector-scalar duality in three dimensions and the magnetic charges of membranes, respectively (see for example [12]). We did not explore the latter two in the present work. In fact, a naive definition of magnetic charge such as  $Tr[Z^a, Z^{a'}]$  for finite matrix configurations vanishes identically.
2. For IIA/M duality, there is a complete dictionary of the correspondence between the spectrum, as well as operators, between IIA D0 brane and M-theory objects, in the Matrix String Theory *a la* DVV [8]. As for IIB/M duality a comprehensive dictionary for spectrum and operators between the two sides remains to be worked out. Moreover, in this work we recovered only part of the  $SL(2, \mathcal{Z})$ -invariant spectrum. A more detailed study to demonstrate the full  $SL(2, \mathcal{Z})$  symmetry is, in principle, possible in the present framework for IIB strings, and we leave it for future research. How to relate this approach to other nonperturbative IIB string theory, such as IIB matrix model [31] or that based on the D-string action [9], is another interesting issue to address.
3. A discrete approach, the string bit model, has been proposed to IIB string theory before [32]. The difference between our quiver mechanics approach

and the string bit model is the latter discretizes the non-linear sigma model for string theory on sites of a lattice, while we deconstruct SYM with part of matter fields living on links. Recently the string bit model gained revived interests [33] in the context of the BMN correspondence [34]. On the other hand, BMN have devised a (massive) matrix model in PP-wave background. How to do orbifolding with this M(atrix) model remains a challenge.

4. Probing spacetime with strings has revealed T-duality of spacetime; i.e. stringy dynamics gives rise to new features of spacetime geometry. In the same spirit, one expects that probing spacetime with membranes would expose new, subtle properties or structures in spacetime geometry too. Actually it has been suggested [13] that M(atrix) Theory compactified on circle, on two- and three-torus are tightly related to each other. We leave the exploration in the framework of quiver mechanical deconstruction to the future.
5. The appearance of non-unitary fields living on links is a generic phenomenon in discrete models. In addition to our previous work [35], a few other authors also paid attention to the effects of non-unitary link fields [36, 37, 38]. Here we would like to emphasize the significant role of non-unitary link fields in dynamics of geometry, which is worthwhile to explore in depth.

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